Calculus Rules For Derivatives

Notation for first derivatives: There are several different ways to represent a first derivative. Recall that the **formal definition of derivative** says: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ and sometimes we see the formula written as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, so either side of either equation could be used as notation for a derivative. With such notation one would want to find the derivative for any point on the curve. In addition, one could see the **alternate form** for a derivative which would be concerned with the result for a specific x value. Thus we could see a derivative with a specified x value in the form $f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ where a would be a given value.

Derivative notation is not limited to the definitions. At times there are other notations that are asking for the same process. One may experience derivative notation such as **differentials**: $\frac{dy}{dx}, \frac{d}{dx}$(expression in terms of x), and $\frac{df}{dx}$ or $\frac{dg}{dx}$ when the function in question is f (x) or g (x) respectively. With this notation if a derivative is needed for a specific point it may be written like $\frac{dy}{dx}\Big|_{x=a}$ or $\frac{df}{dx}\Big|_{x=a}$ where a is a given value. Derivitives may also be taken with respect to other variables such as $f'(t)$ or $\frac{dy}{dt}$. And finally there is the abbreviated notation $D_x[y]$, $D_x[f(x)]$, and so one must understand all the ways in which the first derivative can be represented.

Notation for second derivatives and beyond: These notations are commonly used and as such each student should be well aware of the "language of calculus." Hopefully the table below will help foster that understanding.

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x(y)$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2(y)$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3(y)$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4(y)$
Nth Derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n(y)$

Physics and Notation: On occasion it may be implied that a derivative is needed without actually using the notation. Displacement, velocity, and acceleration would be the prime example where the notation may not as obvious as in other places. Students should understand that the derivative of displacement is velocity, and the second derivative to to displacement (first derivative to velocity) is acceleration.

$S(t) =$		
$S'(t) =$	$V(t)=$	
$S''(t) =$	$V'(t)=$	$A(t)$

Constant Rule: Recall the meaning of a derivative. A derivative is the slope of the tangent line for a given point. If one were to graph a constant by itself the result would be a **horizontal line**. Since horizontal lines have slopes of **zero,** it stands to reason that the derivative of any constant value must also be **zero.** Thus, the derivative of any constant value will be difined as zero.

$$f(x) = a$$
$$f'(x) = 0$$

Trigonometric Functions: After studying the formal definition of derivatives and applying it to the six trigonometry functions one would hope that the derivatives are now memorized. If not here is a chart that should be committed to memory.

Trig Function	Derivative
$f(\theta) = \sin\theta$	$f'(x) = \cos\theta$
$f(\theta) = \cos\theta$	$\frac{d}{d\theta}\left[f(\theta)\right] = -\sin\theta$
$f(\theta) = \tan\theta$	$\frac{df}{d\theta} = \sec^2\theta$
$y = \csc\theta$	$y' = -\csc\theta \cdot \cot\theta$
$y = \sec\theta$	$\frac{dy}{d\theta} = \sec\theta \cdot \tan\theta$
$y = \cot\theta$	$D_\theta(y) = -\csc^2\theta$

Power Rule: The power rule essentially commands that when a term is written in exponential form that the derivative is found by multiplying the base by the original exponent and then reducing the exponent by one.

$$f(x) = x^a$$
$$f'(x) = a\cdot x^{a-1}$$

Examples: Power Rule

Find	Function	Exponential Form	Result
$f'(x)$	$f(x) = x^2 + 3x + 5$		
y'	$y = 5x^3 - 7x^2 + x - 4$		
$\frac{d}{dx}[f(x)]$	$\frac{d}{dx}\left(\frac{-3}{x^4} + \frac{4}{x^2} - \frac{9}{x} + \frac{5}{8}\right)$		
$D_x(y)$	$y = \frac{2}{x^3} - \frac{7}{x} + 3x - 11$		
$f'(x)$	$f(x) = \frac{3}{\left(\sqrt{x} - 4\right)^5}$		
$\frac{dy}{dx}$	$y = \frac{3}{(x-7)^4}$		
V(t)	$S(t) = 22t^2 - 5t + 9$		
$\frac{d^2y}{dx^2}$	$\frac{dy}{dx} = 9x^{\frac{2}{3}} - \frac{1}{2}x^{\frac{2}{5}} + 3x - \frac{11}{17}$		
$\frac{d^3g}{dx^3}$	$g'(x) = 3x^4 - 2x^3 - x^2 + 11x - 3$		

Product Rule: When a function is comprised of the product of two components the derivative can be found as follows:

$f(x) = u\cdot v$ $f'(x) = u\cdot dv + v\cdot du$	$h(x) = f(x)\cdot g(x)$ $h'(x) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

Extending The Product Rule to Three Terms:

$y = h(x)\cdot f(x)\cdot g(x)$

$D_x(y) = \underline{h'(x)}\cdot f(x)\cdot g(x) + h(x)\cdot \underline{f'(x)}\cdot g(x) + h(x)\cdot f(x)\cdot \underline{g'(x)}$

Examples: Product Rule

Find	Function	Final Result
$f'(x)$	$f(x) = 13x^2(x-2)^5$	
y'	$y = 3\pi x\cdot \sin x$	
$\frac{d}{dt}[f(t)]$	$\frac{d}{dt}\left((t-4)^3\cdot(t+9)^7\right)$	
$D_z(y)$	$y = (z+8)^5\cdot(z-11)^3\cdot(z+3)^7$	
$\frac{dy}{dw}$	$y = \sin w\cdot \tan w$	
V(t)	$S(t) = 5t^2(\tan t) - 3t(\sec t) - 11$	
$\frac{d^2y}{dx^2}$	$\frac{dy}{dx} = 9x^{\frac{2}{3}}(x-3)^{\frac{1}{2}}$	
$\frac{d^3h}{dt^3}$	$h''(t) = (t-3)^4\cdot(t+1)^2\cdot(5t-4)$	

Quotient Rule: When a function is comprised of the quotient of two components the derivative can be found as follows:

$f(x) = \dfrac{High}{Low}$ $\dfrac{d}{dx}\left[\dfrac{High}{Low}\right] = \dfrac{Low \cdot d(High) - High \cdot d(Low)}{Low^2}$	$f(x) = \dfrac{f(x)}{g(x)}$ $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

Examples: Quotient Rule

Find	Function	Final Result
$g'(\lambda)$	$g(\lambda) = \dfrac{\lambda}{1 - \sin\lambda}$	
y'	$y = \dfrac{(x-3)^5}{(x+4)^2}$	
$\dfrac{d}{dx}[f(x)]$	$\dfrac{d}{dx}(\tan(x))$	
$D_h(y)$	$y = \left(\dfrac{3h-2}{2h+3}\right) \cdot (5h+1)$	
$\dfrac{dy}{dx}$	$y = \dfrac{\sin x}{1 - \cos x}$	
V(t)	$S(t) = \dfrac{t^2 - 3t + 5}{t+1} + 5t(t+1)^3 + 9$	
$\dfrac{d^2y}{dx^2}$	$\dfrac{dy}{dx} = 3x^2\left(4 - \dfrac{1}{x-2}\right)$	

Chain Rule: When a function has multiple parts, usually with embedded paraentheses, then this rule must be applied. The chain rule is not a rule onto itself, it serves as a universal companion to the already established rules. Think of the parentheses as actual links in a chain that have to be drawn out of the function.

$f(x) = 3(5x^3 - 3x)^6$	$f(x) = \sin^3(5x^2)$
$\frac{d}{dx}\left[3(5x^3 - 3x)^6\right] = 18(5x^3 - 3x)^5 \cdot (15x^2 - 3)$	$\frac{d}{dx}\left[\sin^3(5x^2)\right] = 3 \cdot \sin^2(5x^2) \cdot [\cos(5x^2)] \cdot 10x$
$\frac{d}{dx}\left[3(5x^3 - 3x)^6\right] = 54x^5(5x^2 - 3)^5 \cdot (5x^2 - 1)$	$\frac{d}{dx}\left[\sin^3(5x^2)\right] = 30x \cdot \sin^2(5x^2) \cdot [\cos(5x^2)]$

Examples: Chain Rule

Find	Function	Final Result
$g'(\mu)$	$g(\mu) = 5\tan(3\mu^2)$	
y'	$y = \sin^5(x^4)$	
$\frac{d}{dx}[f(x)]$	$\frac{d}{dx}\left((3x^5 - 8)^{11}\right)$	
$D_h(y)$	$y = 11\cos^5(4x^2 - 5)^3$	
$\frac{d}{dx}[f(x)]$	$\frac{d}{dx}\left((5x^3 + 2)^{10} \cdot (4x^3 - 2)^7\right)$	
V(t)	$S(t) = (4t^3 - 5)^{11} \cdot (5t^2 - 7)^9$	
$\frac{d^2y}{dx^2}$	$\frac{dy}{dx} = 2\sin^2(7x^3) \cdot \cos^4(5x^5)$	
$\frac{dy}{dx}$	$y = \left(\frac{\sin(5h)}{\cos(7h)}\right)$	
$g'(\alpha)$	$g(\alpha) = \frac{(3\alpha^4 - 5)^5}{(2\alpha^5 - 4)^3}$	
$D_h(y)$	$y = \left(\frac{\sin^2(5h^3 - 2)}{\cos^2(7h^2 + 3)}\right)$	