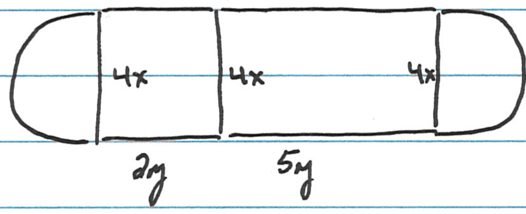
#1 PERIMETER = 840 METERS

ALGEBRAIC EXPRESSION FOR PERIMETER (ACTUAL FENCE NEEDED)



LENGTH : $14y$

WIDTH : $12x$

CIRCLE : $4\pi x$

$P = 12x + 4\pi x + 14y$

$$840 = 12x + 4\pi x + 14y$$

$$\frac{840 - 12x - 4\pi x}{14} = y$$

OBJECTIVE: MAXIMUM AREA

AREA RECTANGLE + AREA CIRCLE

AREA FUNCTION : $7y(4x) + 4\pi x^2$

$$A(x,y) = 28yx + 4\pi x^2$$

$$A(x) = 28\left[\frac{840 - 12x - 4\pi x}{14}\right]x + 4\pi x^2$$

$$A(x) = (1680 - 24x - 8\pi x)x + 4\pi x^2$$

$$A(x) = 1680x - 24x^2 - 8\pi x^2 + 4\pi x^2$$

$$A(x) = 1680x - 24x^2 - 4\pi x^2$$

TAKE DERIVATIVE, SET EQUAL TO ZERO, FIND POSSIBLE MAX/MIN

$$A'(x) = 1680 - 48x - 8\pi x$$

$$0 = 1680 - 48x - 8\pi x$$

$$48x + 8\pi x = 1680$$

$$(48 + 8\pi)x = 1680$$

$$x = 1680/(48 + 8\pi)$$

$$x = 22.97$$

VERIFY MAX/MIN : FIND $A''(22.97)$

$$A''(x) = -48 - 8\pi$$

$A''(22.97)$ = ALWAYS NEG WHICH MEANS CONCAVE DOWN

SO MAX AT $x = 22.97$

MAX AREA = $A(22.97) = 19{,}296.4\ m^2$

#2 CLOSED RECTANGULAR BOX

VOLUME = 1225 IN^3

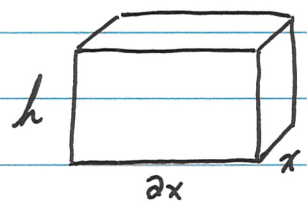
FORMULA FOR BOX: $V = lwh$

FOR PROBLEM $V = x(2x)h$

VOLUME: $1225 = 2x^2 h$

$$\frac{1225}{2x^2} = h$$

OBJECTIVE: MINIMUM SURFACE AREA



SURFACE AREA: $2[2x(x) + xh + 2xh]$

$$S(x,h) = 4x^2 + 2xh + 4xh$$
$$= 4x^2 + 6xh$$
$$S(x) = 4x^2 + 6x\left[\frac{1225}{2x^2}\right]$$

$$S(x) = 4x^2 + \frac{3675}{x}$$

TAKE DERIVATIVE, SET EQUAL TO ZERO, FIND POSSIBLE MAX/MIN

$$S'(x) = 8x - \frac{3675}{x^2}$$

$$0 = 8x - \frac{3675}{x^2}$$

$$\frac{3675}{x^2} = 8x$$

$$\frac{3675}{8} = x^3$$

$$7.7159 = x$$

CHECK FOR MIN AS $S''(x)$ SHOULD BE POSITIVE AT $x = 7.7159$

$$S''(x) = 8 + \frac{7350}{x^3}$$

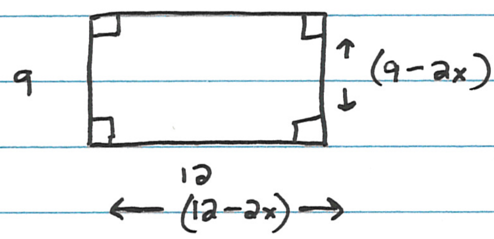
$S''(7.7159) = 24$ SO CONCAVE UP

DIMENSIONS: LENGTH: 15.4318 in

WIDTH: 7.7159 in

HEIGHT: 10.2880 in

#3



$$V(x) = (9-2x)(12-2x)x$$
$$= 108x - 42x^2 + 4x^3$$
$$V(x) = 4x^3 - 42x^2 + 108x$$

OBJECTIVE MAXIMUM VOLUME DIMENSIONS

$$V'(x) = 12x^2 - 84x + 108$$
$$0 = 12x^2 - 84x + 108$$
$$x = \{1.6972, 5.3028\}$$

VERIFY WITH $V''(x)$

$$V''(x) = 24x - 84$$

$V''(1.6972) = -43.267$, CONCAVE DOWN

SO $x = 1.6972$ PRODUCES A MAX

$V''(5.3028) = 43.267$, CONCAVE UP

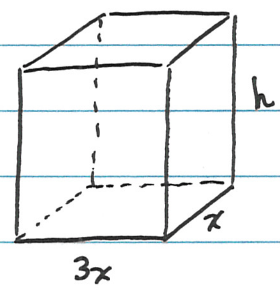
SO $x = 5.3028$ IS EXTRANEOUS

HEIGHT = 1.6972 ft

WIDTH = 5.6056 ft

LENGTH = 8.6056 ft

#4 Build a Box



Height : h

Base width: x

Base length: $3x$

Volume: 50 ft^3 Volume: $3x(x)h$

$$50 = 3x^2h$$

$$\frac{50}{3x^2} = h$$

Objective: minimize cost of making box (consider surface area)

Cost function: $C(x,h) = \$10\,\underbrace{[2(x)(3x)]}_{\text{Top / Bottom}} + \$6\,\underbrace{[2(x)(h)] + \$6[2(3x)(h)]}_{\text{Sides}}$

$$C(x) = 60x^2 + 12x\left[\frac{50}{3x^2}\right] + 36x\left[\frac{50}{3x^2}\right]$$

$$C(x) = 60x^2 + \frac{800}{x}$$

Take derivative, set equal to zero, find possible max/min values

$$C'(x) = 120x - \frac{800}{x^2}$$

$$0 = 120x - \frac{800}{x^2}$$

$$\frac{800}{x^2} = 120x$$

$$6\tfrac{2}{3} = x^3$$

$$1.8821 = x$$

Verify minimum by checking that $C''(1.8821)$ is concave up.

$$C''(x) = 120 + \frac{1600}{x^3}$$

$C''(1.8821) = 359.9893$ Positive result implies concave up

So minimum at $x = 1.8821$

Dimensions

Width: 1.8821 ft

Length: 5.6463 ft

Height: 4.7050 ft

#5 CYLINDRICAL CAN

VOLUME = 3.5 LITERS   VOLUME = $\pi r^2 h$

* $1 cm^3 = 1 mL$

VOLUME = $3{,}500\ cm^3$

$$3{,}500 = \pi r^2 h$$

OBJECTIVE: MINIMIZE (SURFACE AREA) REPORT DIMENSIONS

$$S_A = 2\pi r^2 + 2\pi r h$$

$$S(r) = 2\pi r^2 + 2\pi r\left[\frac{3{,}500}{\pi r^2}\right]$$
$$S(r) = 2\pi r^2 + \frac{7000}{r}$$

TO FIND POSSIBLE MAX/MIN, TAKE DERIVATIVE - SET EQUAL TO ZERO

$$S'(r) = 4\pi r - \frac{7000}{r^2}$$

$$0 = 4\pi r - \frac{7000}{r^2}$$
$$\frac{7000}{r^2} = 4\pi r$$
$$\frac{7000}{4\pi} = r^3$$
$$8.2280\,cm = r$$

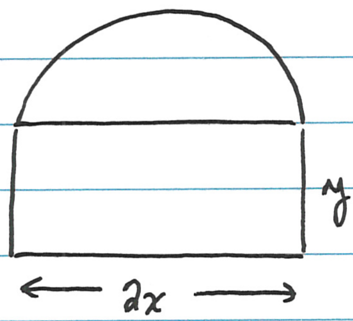
VERIFY MINIMUM BY CHECKING $S''(r)$ IS POSITIVE AT

$$S''(r) = 4\pi + \frac{14000}{r^3}$$
$S''(8.2280) = 37.6994$ SO CONCAVE UP AT $r = 8.2280$

DIMENSIONS: RADIUS = 8.2280 cm

HEIGHT = 16.4562 cm

#6 Norman Window



Exterior Perimeter = 52 ft

Algebraic Expression:

Perimeter: $2x + 2y + (2\pi x)\cdot\frac{1}{2}$

$= 2x + 2y + \pi x.$

Perimeter: $52 = 2x + 2y + \pi x$

$\frac{52 - 2x - \pi x}{2} = y$

Objective: Dimensions of maximum area

Area of Window: $A(x,y) = 2xy + \frac{1}{2}\pi x^2$

$$A(x) = 2x\left[\frac{52 - 2x - \pi x}{2}\right] + \frac{1}{2}\pi x^2$$

$$A(x) = 52x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$A(x) = 52x - 2x^2 - \frac{1}{2}\pi x^2$$

Take derivative, set equal to zero, find possible max/min

$A'(x) = 52 - 4x - \pi x$

$0 = 52 - 4x - \pi x$

$(4+\pi)x = 52$

$x = 52/4+\pi \approx 7.2813$

Verify max/min by checking $A''(7.2813)$

$A''(x) = -4 - \pi$, alway negative $\Rightarrow$ concave down

$A''(7.2813)$ is negative so maximum at $x = \frac{52}{4+\pi} \approx 7.2813$

Dimensions Length $2x = 14.5626$

Width $y = 7.2813$

#7 FIND PTS ON GRAPH $y = x^2 - 6x + 11$ CLOSEST TO $(3,7)$

THIS SOUNDS VERY MUCH LIKE A DISTANCE.

CONSIDER $D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

AND $y = x^2 - 6x + 11$ REPRESENTED AS ORDERED PAIR $(x, x^2 - 6x + 11)$
ALONG WITH GIVEN POINT $(3,7)$

SO $D^2 = (x-3)^2 + (x^2 - 6x + 11 - 7)^2$

LET $D^2 = f(x) = (x-3)^2 + (x^2 - 6x + 4)^2$

ONE MAY USE THIS TECHNIQUE TO SIMPLIFY THE EQUATION BECAUSE THE MINIMUM RADICANDS (AS PRESENTED WITH THIS EQUATION) WOULD PERTAIN TO THE MINIMUM RADICALS IF ONE WERE TO USE THE MORE COMPLICATED $D = [(x-3)^2 + (x^2 - 6x + 4)]^{1/2}$

OBJECTIVE FIND MINIMUM FOR $f(x)$ AND REPORT (x, y) FROM $y = x^2 - 6x + 11$

$$f(x) = (x-3)^2 + (x^2 - 6x + 4)^2$$

TAKE DERIVATIVE, SET EQUAL TO ZERO, FIND POSSIBLE MAX/MIN VALUES

$$\begin{aligned} f'(x) &= 2(x-3) + 2(x^2 - 6x + 4)(2x - 6) \\ &= 2(x-3)\left[1 + 2x^2 - 12x + 8\right] \\ 0 &= 2(x-3)(2x^2 - 12x + 9) \\ x &= \{.8787, 3, 5.1213\} \end{aligned}$$

VERIFY RESULTS USING $f''(x)$ TO CHECK CONCAVITY

$f''(x) = 12x^2 - 72x + 90$

$f''(.8787) \doteq 35.999 \Rightarrow$ CONCAVE UP, MIN AT $x = .8787$

$f''(3) = -18 \Rightarrow$ CONCAVE DOWN, MAX AT $x = 3$

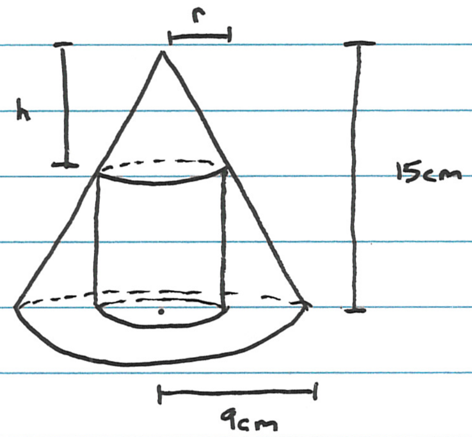
$f''(5.1213) \doteq 35.999 \Rightarrow$ CONCAVE UP, MIN AT $x = 5.1213$

SO (x, y) POINTS ON GRAPH WITH SHORTEST DISTANCE TO $(3,7)$

$(.8787, 6.4999)$

$(5.1213, 6.4999)$

#8



$$\frac{r}{h} = \frac{9}{15}$$

$$r = \frac{9}{15}h$$

Volume for Cylinder $= \pi r^2 H$

* r is same as radius for little cone

but H is 15 minus h for little cone

$H = 15 - h$

$$V = \pi r^2 (15 - h)$$

$$V(h) = \pi \left[\tfrac{9}{15}h\right]^2 (15 - h)$$

$$V(h) = \frac{81}{225}\pi h^2 (15 - h)$$

$$V(h) = \frac{81}{15}\pi h^2 - \frac{81}{225}\pi h^3$$

This problem could also be set up using $V(r)$

Objective: Find cylinder with maximum volume

$$V(h) = 81/15\, \pi h^2 - 81/225\, \pi h^3$$

Take derivative, set equal to zero, find possible max/min

$$V'(h) = \frac{162}{15}\pi h - \frac{243}{225}\pi h^2$$

$$0 = \frac{81}{15}\pi h \left(2 - \frac{1}{5}h\right)$$

$$h = \{0, 10\}$$

Verify max/min using $V''(h)$

$$V''(h) = \frac{162}{15}\pi - \frac{486}{225}\pi h$$

$V''(0) = \frac{162}{15}\pi$, Positive $\Rightarrow$ Concave up, so minimum at $h = 0$

$V''(10) = \frac{-54}{5}\pi$, Negative $\Rightarrow$ Concave down, so max at $h =$

Dimensions for cylinder

Height: $h = 10$

Radius: $r = 6$